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A PERSPECTIVE AND PROPOSAL FOR THE INITIAL TRANSIENT PROBLEM IN--ETC(U)
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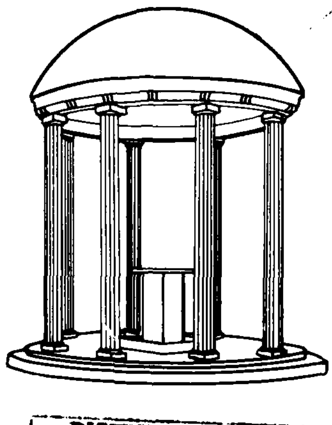

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A PERSPECTIVE AND PROPOSAL FOR THE INITIAL TRANSIENT PROBLEM IN SIMULATION

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and Systems Analysis

University of North Carolina at Chapel Hill

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These remarks were prepared for the invited panel discussion on the initial transient problem in steady-state simulation at the ORSA/TIMS Meeting in Toronto Canada, May 4-6, 1981.

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Abstract

This paper presents a perspective on the initial transient problem in steady-state simulation. In particular, it enumerates five generally accepted facts: 1) Conditions prevailing at the beginning of a simulation influence sample paths. 2) The extent of influence is a function of the strength of autocorrelation. 3) Some initial conditions are less detrimental than others are. 4) Truncation reduces bias but usually increases variance. 5) So far no complete solution exists. The remainder of the paper describes a proposal for solving the problem. It relies on the relatively weak assumption that the conditional means in a stochastic process of interest are related linearly. An estimator of the steady-state mean is described which has considerably less bias than one can achieve via conventional truncation. An interval estimator is also described which follows from standard regression theory. A test for residual bias is presented which enables a user to judge whether or not sample data meet the minimal requirements for the proposed technique to apply. A second test allows a user to judge whether or not a more efficient estimation technique can be used.

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1. Perspective

As you the audience know, the chairman's charge to the invited speakers today is to provide an up-to-date perspective on the problem of the initial transient in steady-state simulations. Although each speaker has his own distinct view of this problem, one hopes that all can agree on a skeletal characterization of it. In particular, in a simulation:

1. The initial conditions that prevail at the beginning of a run influence the sample path that each stochastic process, represented in the simulation, follows.
2. The extent to which the initial conditions affect a stochastic process at a given point in a run is a function of the degree of autocorrelation in the process.
3. Some intial conditions influence a stochastic process at a given point in a run to a lesser extent than other initial conditions do.
4. Truncation of observations near the beginning of a run reduces the bias in the sample mean as an estimator of the steady-state mean but generally increases its variance.
5. No completely satisfactory procedure for resolving the problem has appeared yet.

In support of Point 5 note that during the past year three of the panelists have proposed solutions. See Adlakha and Fishman (1980), Kelton (1980) and Schruben (1979, 1980). However, I doubt anyone would claim a definitive solution.

Perhaps, one of the reasons that a solution has alluded us is that while the published characterization of the underlying structures in

the stochastic processes of interest has enabled us to conceptualize the problem it has been too narrow to allow a satisfactory solution for a wide range of cases. In particular, I refer to the first-order autoregressive representation used in Fishman (1972) and more recently in Kelton (1980) and Schruben (1980). To overcome this inadequacy we describe a generalization of that model and show how it leads directly to an estimator of the steady-state mean that is relatively free of contamination from an initial transient. The remainder of this section provides a concise description of the essential points of the proposal. The Appendix contains a more detailed exposition.

2. Proposal

Let I be the initial conditions that prevail in a simulation and let $X_1^{(i)},\ldots,X_{n_i}^{(i)}$ be the sample record collected on the ith of $2m'$ independent replications. The objective is to estimate the steady-state mean

$$\mu = \lim_{j\to\infty} E\, X_j^{(i)} \qquad i=1,\ldots,m=2m'.$$

For convenience of exposition we take

$$\begin{aligned} n_{2i-1} &= N_1 + k \\ n_{2i} &= N_2 + k \qquad N_1 < N_2 \qquad i=1,\ldots,m'. \end{aligned}$$

The more general case of arbitrary $n_1\ldots,n_m$ is described in the Appendix.

We now impose a restriction on $\{X_j^{(i)};\ j=1,2,\ldots\}$ that, while weak, has relatively profound implications for the estimation of μ. If the regression of $X_{j-1}, X_{j-2}, \ldots$ on X_j is linear and of the form

$$E\,(X_j \mid X_1,\ldots,X_{j-1},\ I) = b + \sum_{s=1}^{p} a_s\, X_{j-s} \qquad (*)$$

then

$$\hat{\mu}_{k,n_i}^{(i)} = \frac{1}{n_i-k} \sum_{j=k+1}^{n_i} X_j^{(i)} \qquad p < k < n_i \qquad i=1,\ldots,m$$

has expectation

$$E\,\hat{\mu}_{k,n_i}^{(i)} = \mu + \frac{g_k}{n_i-k} - \frac{g_{n_i}}{n_i-k} \qquad (**)$$

where

$$g_t = 0(\gamma^{t-p}) \qquad 0 < \gamma < 1,\ \ p < t.$$

Here k is the *truncation* parameter. More importantly, the estimator

$$\tilde{\mu} = \frac{2}{m(N_2-N_1)} \sum_{i=1}^{m'} (N_2\, \hat{\mu}_{k,N_2+k}^{(2i)} - N_1\, \hat{\mu}_{N_1+k}^{(2i-1)}) \qquad (***)$$

has expectation

$$E\,\tilde{\mu} = \mu + (g_{N_1+k} - g_{N_2+k}) / (N_2 - N_1)$$

$$= \mu + 0\,(\gamma^{N_1+k-p}/(N_2-N_1)).$$

Observe that whereas the *within-replication* truncated sample means (**) have bias $O(\gamma^{k-p}/N_1)$ the new *across-replication* estimator (***) dilutes this bias to $O(\gamma^{N_1+k-p}/(N_2-N_1))$, provided that $N_2 \geq 2N_1$. Moreover,

$$\hat{\text{var}}\ \tilde{\mu} = 2\hat{\sigma}^2 (N_1 + N_2) / m (N_2 - N_1)^2,$$

where $\hat{\sigma}^2$ is defined in (19) in the Appendix, gives an asymptotically $(N_1 \to \infty)$ unbiased estimator of var $\tilde{\mu}$. Provided that $\{\hat{\mu}^{(i)}_{k,n_i};\ i = 1,\ldots,m\}$ are normal, one can treat $(\tilde{\mu}-\mu) / \sqrt{\hat{\text{var}}\ \tilde{\mu}}$ as t distributed with $m-2$ degrees of freedom.

Let us now concentrate on the plausibility of (*) as an underlying characterization. Clearly, (*) hold for autoregressive processes with normal disturbances. It also holds for a variety of stationary sequences of nonnegative random variables with gamma marginal distributions. See Lewis (1979). It also holdsfor Markov chains. More generally, provided that the expectation on the left is bounded for all j, one can treat the right side of (*) as an approximation to the left side that either becomes exact for some p or whose error of approximation can be restricted by making p suitably large. If the analogy with fitting a pth-order polynomial is kept in mind, p needs to accomodate the smoothness requirements of $E(X_j \mid X_1,\ldots,X_{j-1}, I)$ for $j = p + 1, p + 2, \ldots$.

To assure oneself of the relative insignificance of the bias in $\tilde{\mu}$, a test based on $\{X^{(2i-1)}_{N_1+k},\ X^{(2i)}_{N_2+k};\ i = 1,\ldots,m'\}$ is provided in the

Appendix. Briefly, if $E\, X^{(2i-1)}_{N_1+k} - E\, X^{(2i)}_{N_2+k} \approx 0$, then $g_{N_1+k} \approx 0$.

Although it is difficult to choose a k such that $g_k = 0$, one would hope to be capable of choosing $N_1 + k$ so that $X^{(2i-1)}_{N_1+k}$, the last observation collected in each other the shorter replications, is unbiased. If no significance is found then $\tilde{\mu}$ is the best linear unbiased estimator of μ based on $\{\hat{\mu}^{(2i-1)}_{k,N_1+k},\ \hat{\mu}^{(2i)}_{k,N_2+k};\ i = 1,\ldots,m'\}$.

Occasionally one may pick k sufficiently large so that $g_k/N_1\mu$ is relatively incidental. In this case the estimator

$$\hat{u} = \frac{2\sum_{i=1}^{m'} \left(\sqrt{N_1}\ \hat{\mu}^{(2i-1)}_{k,N_1+k} + \sqrt{N_2}\ \hat{\mu}^{(2i)}_{k,N_2+k}\right)}{m\left(\sqrt{N_1} + \sqrt{N_2}\right)}$$

gives smaller variance than $\tilde{\mu}$. The Appendix provides a method of computing a confidence interval for $g_k/N_1\mu$.

Figure 1 shows the essential steps to follow and procedure M in the Appendix contains all required computational expressions. We remark that the choice of two sample sizes $N_1 + k$ and $N_2 + k$ leads to considerable convenience and simplication with regard to estimating μ.

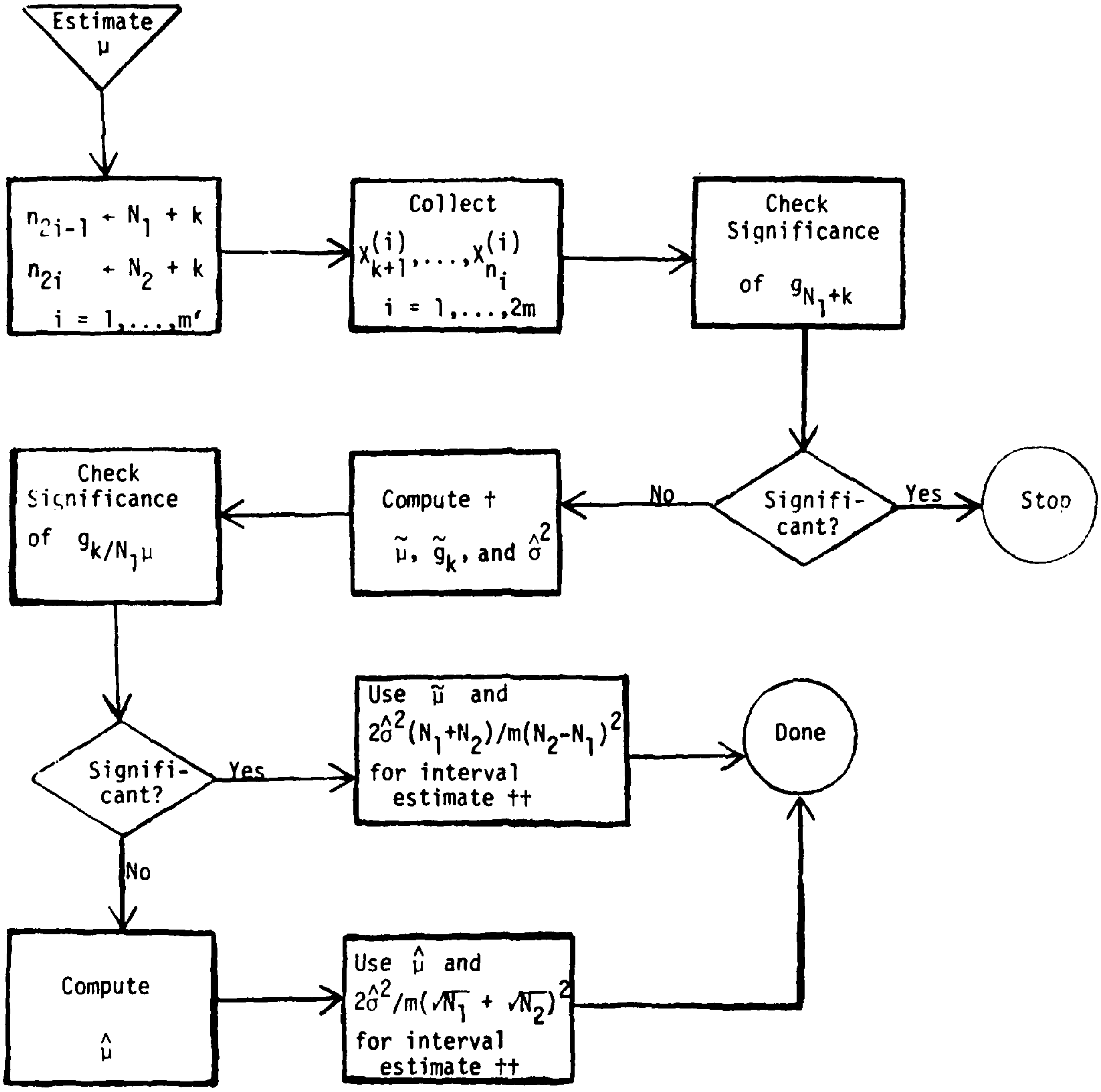

Figure 1. Estimation of μ

† See (14) in Appendix for $\tilde{g}_k$.

†† Use t distribution with $m-2$ degrees of freedom.

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APPENDIX

1. Preliminary Definitions and Assumptions

Let I denote the conditions that prevail at the beginning of a simulation. Let $\mu_j = E(X_j|I)$ $j = 1,2,\ldots$ denote the conditional mean of observation j and $\mu = E X_j$, the steady-state mean. Assume that

$$\lim_{j\to\infty} \mu_j = \mu \tag{1}$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{n} (\mu_j-\mu) = c \qquad |c| < \infty . \tag{2}$$

These relatively weak assumptions are necessary if we hope to deduce a method of estimating μ when the influence of I is present.

Let us now assume that the *regression* of $X_{j-1}, X_{j-2},\ldots$ on X_j is linear and has the form

$$E(X_j|X_1,\ldots,X_{j-1},I) = b + \sum_{s=1}^{p} a_s X_{j-s} \tag{3}$$

$$j = p+1,\ p+2,\ldots .$$

Models of this type arise in conventional autoregressive analysis and in representation of sequences of nonnegative dependent random variables. See Lewis (1979). After integration over the domain of $X_1,\ldots,X_{j-1}$ one has

$$\mu_j = b + \sum_{s=1}^{p} a_s \mu_{j-s}$$

where, using (1),

$$b = \mu\,(1 - \sum_{s=1}^{p} a_s). \tag{4}$$

Let us define $\omega_j = \mu_j - \mu$. If $\sum_{s=1}^{p} a_s \neq 1$, then (2) gives

$$c = \frac{\sum_{j=1}^{p} \omega_j \sum_{s=0}^{p-j} a_s}{\sum_{s=0}^{p} a_s} \qquad a_0 \equiv -1 . \qquad (5)$$

Note the dependence on the first p conditional means $\mu_1, \ldots, \mu_p$ so that c is a function of I.

Consider an estimator of μ of the form

$$\hat{\mu}_{k,n} = \frac{1}{n-k} \sum_{j=k+1}^{n} X_j \qquad p \le k < n . \qquad (6)$$

The quantity k denotes the number of observations truncated from the sample at the beginning of the run. We can use

$$\omega_j = \sum_{s=1}^{p} a_s \, \omega_{j-s} \qquad j > p \qquad (7)$$

to characterize the bias in (6). In particular, observe that

$$E \, \hat{\mu}_{k,n} = \mu + \frac{g_k - g_n}{n-k} \qquad (8)$$

where

$$g_t = \frac{\sum_{s=1}^{p} a_s \sum_{j=t+1-s}^{t} \omega_j}{1 - \sum_{s=1}^{p} a_s} . \qquad (9)$$

Note that the bias in $\hat{\mu}_{k,n}$ is a function of $\mu_{k+1-p}, \ldots, \mu_k$ and $\mu_{n+1-p}, \ldots, \mu_n$ which are related to the beginning and end, respectively,

of the sample record $X_{k+1},\ldots,X_n$. We can characterize this bias more precisely by using (7). The solution to this pth-order difference equation has the form

$$\omega_j = \sum_{i=1}^{p} c_i \, \beta_i^j \tag{10}$$

where

$$|\beta_i| < |\beta_{i+1}| < 1 \qquad i=1,\ldots,p-1$$

so that

$$g_t = O(|\beta_p|^{t-p}) \qquad t > p \tag{11}$$

and

$$E(\hat{\mu}_{k,n} - \mu) = \frac{1}{n-k}\left[O(|\beta_p|^{k-p}) - O(|\beta_p|^{n-p})\right] .$$

Since $g_n/g_k \approx 0$ when $n >> k$, we concern ourselves principally with the more serious bias $g_k/(n-k)$. In particular, we now turn to the estimation of μ free of the contamination of $g_k/(n-k)$.

2. Estimation of μ

Consider m independent replications of the simulation, each with initial conditions I. For replication i let n denote the number of observations collected and $\hat{\mu}_{k,n_i}^{(i)}$ the sample mean based on observations $k + 1$ through n_i with $k < n_1 \le n_j$ $j=2,\ldots,m$ and where the inequality holds for at least one n_j. Then

$$E\ \hat{\mu}^{(i)}_{k,n_i} = \mu + \frac{g_k}{n_i - k} - \frac{g_{n_i}}{n_i - k} \quad . \tag{12}$$

Let

$$f_i = n_i - k \qquad\qquad i = 1,\ldots,m$$

$$D_m = \sum_{i=1}^{m} f_i \sum_{j=1}^{m} f_j^{-1} - m^2 \ . \tag{13}$$

Consider the quantities

$$\begin{aligned} \tilde{\mu} &= D_m^{-1} \sum_{i=1}^{m} \left(f_i \sum_{j=1}^{m} f_j^{-1} - m\right) \hat{\mu}^{(i)}_{k,n_i} \\ \tilde{g}_k &= D_m^{-1} \sum_{i=1}^{m} \left(\sum_{j=1}^{m} f_j - m\, f_i\right) \hat{\mu}^{(i)}_{k,n_i} \quad . \end{aligned} \tag{14}$$

Then one has

$$\begin{aligned} E\ \tilde{\mu} &= \mu - A_m \\ E\ g_k &= g_k - B_m \end{aligned} \tag{15}$$

where

$$\begin{aligned} A_m &= D_m^{-1} \sum_{i=1}^{m} g_{n_i} \left(m\, f_i^{-1} - \sum_{j=1}^{m} f_j^{-1}\right) \\ B_m &= D_m^{-1} \sum_{i=1}^{m} f_i\, g_{n_i} \left(m\, f_i^{-1} - \sum_{j=1}^{m} f_j^{-1}\right) \ . \end{aligned} \tag{16}$$

Note that A_m and B_m contain no term in g_k , thus eliminating the bias in $\tilde{\mu}$ and $\tilde{g}_k$ due to this term.

One additional assumption is needed. Suppose that

$$\lim_{n_1 \to \infty} (n_i - k) \operatorname{var} \hat{\mu}^{(i)}_{k,n_i} = \sigma^2 < \infty \quad . \tag{17}$$

Then to order $1/mn_1$ we have

$$\begin{aligned}
\operatorname{var} \tilde{\mu} &= \sigma^2 D_m^{-1} \sum_{i=1}^{m} f_i^{-1} \\
\operatorname{var} \tilde{g}_k &= \sigma^2 D_m^{-1} \sum_{i=1}^{m} f_i \\
\operatorname{cov} (\tilde{\mu}, \tilde{g}_k) &= -m \, \sigma^2 D_m^{-1}
\end{aligned} \tag{18}$$

$$\operatorname{corr} (\tilde{\mu}, \tilde{g}_k) = -m \Big/ \sqrt{\sum_{i=1}^{m} f_i \sum_{i=1}^{m} f_i^{-1}} \quad .$$

Now, as $n_1 \to \infty$ $\tilde{\mu}$ and $\tilde{g}_k$ converge to the generalized least-squares estimators of μ and g_k, respectively, which, in addition to being unbiased, have minimal variance among all linear estimators. As an estimator of σ^2 one has

$$\hat{\sigma}^2 = \frac{1}{m-2} \sum_{i=1}^{m} f_i \left(\hat{\mu}^{(i)}_{k,n_i} - \tilde{\mu} - \tilde{g}_k f_i^{-1}\right)^2 \tag{19}$$

which is asymptotically $(n_1 \to \infty)$ unbiased and for which $(m-2)\,\hat{\sigma}^2/\sigma^2$ has the chi-squared distribution with $m - 2$ degrees of freedom, provided that the $\hat{\mu}^{(i)}_{k,n_i}$ converge to normality. Then $(\tilde{\mu}-\mu)/(\hat{\sigma}^2 D_m^{-1} \sum_{i=1}^{m} f_i^{-1})^{\frac{1}{2}}$ has the t distribution with m-2 degrees of freedom, as does $(\tilde{g}_k - g_k)/(\hat{\sigma}^2 D_m^{-1} \sum_{i=1}^{m} f_i)^{\frac{1}{2}}$.

To illustrate the estimation method consider the case of m even and

$$n_{2i-1} = N_1 + k$$

$$n_{2i} = N_2 + k \qquad N_1 < N_2 \qquad i=1,\dots,m/2 \qquad (20)$$

$$\sum_{i=1}^{m} f_i = m\,(N_1 + N_2)/2, \qquad \sum_{i=1}^{m} f_i^{-1} = m\,(N_1 + N_2)/2N_1N_2$$

$$D_m = m^2\,(N_2 - N_1)^2/4N_1N_2$$

so that to order $1/N_1$

$$\operatorname{var}\tilde{\mu} = 2\sigma^2\,(N_1 + N_2)/\,m\,(N_2 - N_1)^2$$

$$\operatorname{var}\tilde{g}_k = 2\sigma^2\,N\,N_2\,(N_1 + N_2)/\,m\,(N_2 - N_1)^2 \qquad (21)$$

$$= N_1N_2 \operatorname{var}\tilde{\mu}$$

$$\operatorname{cov}(\tilde{\mu},\tilde{g}_k) = -\,4\,N_1N_2\sigma^2/m(N_2 - N_1)^2$$

$$\operatorname{corr}(\tilde{\mu},\tilde{g}_k) = -\,\sqrt{N_1N_2}\,/\,(N_1 + N_2) = -\sqrt{N_2/N_1}\,/\,(1 + N_2/N_1)\,.$$

Also, as $N_1 \to \infty$ the distributions of $(\tilde{\mu}-\mu)\,(N_2 - N_1)\sqrt{m/\hat{\sigma}^2(N_1 + N_2)}$

and $(\tilde{g}_k - g_k)(N_2 - N_1)\sqrt{m/2\hat{\sigma}^2\,N_1N_2(N_1 + N_2)}$ converge to the t distribution with $m-2$ degrees of freedom. Observe that a large difference $N_2 - N_1$, is most desirable. Later sections show that the sampling plan (20) offers many conveniences for estimation and hypothesis testing.

3. Relative Importance of $g_k/(n_1 - k)$

The aforementioned results for $\tilde{\mu}$ and $\tilde{g}_k$ follow directly from the assumptions of a linear regression in (3), a conventional form of convergence in (17) and, to a lesser extent, the asymptotic normality of $\hat{\mu}^{(i)}_{k,n_i}$ for $i = 1,\ldots,m$. Reflecting on the method for a moment, one quickly notes that k does not serve the role of the conventional truncation parameter. We merely require k to exceed p, the order of the difference equation in (3). The value of p depends on the smoothness of μ_j as a function of j. Generally, a choice of, say, $k = 10$ accomodates a commonly encountered range of cases and a choice of $k = 100$ probably includes all but the most exceptional cases.

Suppose that for the selected k $g_k/(n_1 - k)\mu \approx 0$. *Then using*

$$\hat{\mu} = \frac{\sum_{i=1}^{m} f_i^{\frac{1}{2}} \hat{\mu}^{(i)}_{k,n_i}}{\sum_{i=1}^{m} f_i^{\frac{1}{2}}}\,, \tag{22}$$

instead of $\tilde{\mu}$, gives

$$\operatorname{var} \hat{\mu} = m\sigma^2 / \left(\sum_{i=1}^{m} f_i^{\frac{1}{2}}\right)^2 \le \operatorname{var} \tilde{\mu}\,.$$

To take advantage of this opportunity for greater accuracy one can test $\rho = g_k/(n_1-k)\mu$ for significance. For example, suppose we want to test the hypothesis that $|\rho|$ does not exceed a preassigned tolerance $\delta > 0$ significantly and that an investigator is willing to tolerate, say, $\delta = 0.01$ or 0.05 by way of relative error. By using the

method of Fieller (1954) for ratio estimates one can make the confidence statement

$$\mathrm{pr}\ (f_1^{-1}\ r_1 \le \rho \le f_1^{-1}\ r_2) = 1 - \alpha \tag{23}$$

where

$$r_i = \frac{\tilde{g}_k \tilde{\mu} - hm + (-1)^i\ \gamma}{\tilde{\mu}^2 - h \sum_{j=1}^{m} f_j^{-1}} \qquad i = 1,2 \tag{24a}$$

$$\gamma^2 = (\tilde{g}_k\ \tilde{\mu}\ - h\ m)^2 - (\tilde{\mu}^2 - h \sum_{j=1}^{m} f_j^{-1})\ (\tilde{g}_k^2 - h \sum_{j=1}^{m} f_j) \tag{24b}$$

$$h = \hat{\sigma}^2\ D_m^{-1}\ t_{m-2}^2\ (\alpha) \tag{24c}$$

and $t_{m-2}\ (\alpha)$ is the $1 - \alpha/2$ quantile of the t distribution with $m - 2$ degrees of freedom, provided that $\gamma^2 \ge 0$. If the interval $[f_1^{-1}\ r_1,\ f_1^{-1}\ r_2]$ is relatively small, then one may want to use (22) instead of (6) to achieve a smaller variance. For example, if $|r_1| \le \delta\ f_1$ and $|r_2| \le \delta\ f_1$ one may elect to use (22).

4. Testing $\mu_{N_1+k} - \mu = 0$

A remaining issue in need of attention is the implicit assertion that $g_{n_1}/g_k \approx 0$. To check the credibility of the assumption we test the more inclusive hypothesis: $\mu_{n_1} - \mu = 0$, since we also want at least

some observations within a replication to be relatively unbiased. At least one check on the adequacy of this hypothesis seems possible. We describe the check for the case in (20). Let $X^{(2i-1)}_{N_1+k}$ and $X^{(2i)}_{N_2+k}$ denote the last observations collected on replications $2i-1$ and $2i$, respectively and let $m' = m/2$ and

$$Y_i = X^{(2i-1)}_{N_1+k} - X^{(2i)}_{N_2+k} \qquad i = 1,\ldots,m' . \tag{25}$$

If N_1 is sufficiently large so that one can regard $\mu_{N_1+k} - \mu = 0$, then $\bar{Y}_{m'} / \sqrt{s^2(Y)/m'}$, where

$$\bar{Y}_{m'} = \frac{1}{m'} \sum_{i=1}^{m'} Y_i \tag{26a}$$

$$s^2(Y) = \frac{1}{m'-1} \sum_{i=1}^{m'} (Y_i - \bar{Y}_{m'})^2 , \tag{26b}$$

has the t distribution with $m'-1$ degrees of freedom. One can test the hypothesis at a prespecified level α or alternatively compute the P-value

$$P_1 = pr\,(T_{m'-1} \geq |\bar{Y}_{m'}| / \sqrt{s^2(Y)/m'}\,) \tag{27}$$

where $T_{m'-1}$ has the t distribution with $m'-1$ degrees of freedom. Small values of P_1 lower the credibility of the hypothesis.

5. Testing $N_1 \operatorname{var} \hat{\mu}^{(2i-1)}_{k,N_1+k} = N_2 \operatorname{var} \hat{\mu}^{(2i)}_{k,N_2+k} = \sigma^2$

If $g_{n_1}/(n_1 - k)\,\mu$ is found to be significant then one may be suspicious of the assumption that $\operatorname{var} \hat{\mu}^{(i)}_{k,n_i} \approx \sigma^2/(n_i - k)$ $i=1,\ldots,m$.

To test this hypothesis when the sampling plan (20) holds, one can use the statistic

$$F_{m'-1,m'-1} = \frac{N_1 \sum_{i=1}^{m'} (\hat{\mu}_{k,N_1}^{(2i-1)} + k - Q_1)^2}{N_2 \sum_{i=1}^{m'} (\hat{\mu}_{k,N_2}^{(2i)} + k - Q_2)^2} , \tag{28}$$

where

$$Q_1 = \frac{1}{m'} \sum_{i=1}^{m'} \hat{\mu}_{k,N_1}^{(2i)} + k \tag{29a}$$

$$Q_2 = \frac{1}{m'} \sum_{i=1}^{m'} \hat{\mu}_{k,N_2}^{(2i)} + k \quad . \tag{29b}$$

If $N_1 \operatorname{var} \hat{\mu}_{k,N_1}^{(2i-1)} + k = N_2 \operatorname{var} \hat{\mu}_{k,N_2}^{(2i)} + k = \sigma^2 \qquad i = 1,\ldots,m'$,

$F_{m'-1,m'-1}$ has the F distribution with $m'-1$ and $m'-1$ degrees of freedom.

In the absence of additional information one might incline to use a two-tail test. However, the observation that the initial conditions I are more restrictive of the variation in $\hat{\mu}_{k,N_1}^{(2i-1)} + k$ than they are of the variation in $\hat{\mu}_{k,N_2}^{(2i)} + k$ leads one to consider a one-tail test. Let $F(\beta)$ be the β-quantile of the F distribution with $m'-1$ and $m'-1$ degrees of freedom. If $F(\beta) \le F_{m'-1,m'-1}$ we accept the hypothesis at the $1 - \beta$ level.

If the hypothesis of equality is rejected then the variance expression (19) does not apply. Also, the use of the t distribution

with m-2 degree of freedom no longer applies exactly. When (20) is used one has

$$\hat{\mu} = \frac{N_2}{N_2-N_1} Q_2 - \frac{N_1}{N_2-N_1} Q_1 \quad . \tag{30}$$

Let

$$\sigma_1^2 = \mathrm{var}\ \hat{\mu}^{(2i-1)}_{k,N_1+k}$$

$$\sigma_2^2 = \mathrm{var}\ \hat{\mu}^{(2i)}_{k,N_2+k} \qquad i=1,\ldots,m'$$

so that

$$\mathrm{var}\ \tilde{\mu} = \frac{2}{m(N_2-N_1)^2} (N_1^2\ \sigma_1^2 + N_2^2\ \sigma_2^2) \quad .$$

As unbiased estimates of σ_1^2 and σ_2^2 one has, respectively,

$$s_1^2 = \frac{1}{m'-1} \sum_{i=1}^{m'} (\hat{\mu}^{(2i-1)}_{k,N_1+k} - Q_1)^2 \tag{31a}$$

$$s_2^2 = \frac{1}{m'-1} \sum_{i=1}^{m'} (\hat{\mu}^{(2i)}_{k,N_2+k} - Q_2)^2 \quad . \tag{31b}$$

Then it is common practice to treat

$$G_{m'-1} = \frac{1}{N_2-N_1} \sqrt{\frac{2}{m} (N_1^2\ s_1^2 + N_2^2\ s_2^2)} \tag{32}$$

as a t distributed random variable with $m'-1$ degrees of freedom, although this treatment is at best an approximation.

Procedure M

Definitions: $t_r(\beta)$ = 1 - $\beta/2$ quantile of t distribution with r degrees of freedom. δ = tolerable relative bias.

Given: k, $N_1 < N_2$, m (even), $m' = m/2$, $t_{m'-1}(2\alpha)$, $t_{m-2}(\alpha)$ and δ.

Collect: $X_{k+1}^{(2i-1)},\ldots,X_{N_1+k}^{(2i-1)},X_{k+1}^{(2i)},\ldots,X_{N_2+k}^{(2i)}$ $\quad i = 1,\ldots,m'$.

Test: $\mu_{N_1+k} - \mu = 0$.

1. $\bar{Z} \leftarrow \frac{1}{m'} \sum_{i=1}^{m'} [X_{N_1+k}^{(2i-1)} - X_{N_2+k}^{(2i)}]$.

2. $s^2 \leftarrow \frac{1}{m'(m'-1)} \sum_{i=1}^{m'} [X_{N_1+k}^{(2i-1)} - X_{N_2+k}^{(2i)} - \bar{Z}]^2$.

3. $t' \leftarrow \bar{Z} / \sqrt{s^2}$.

4. If $t' > t_{m'-1}$ (2α), return indicating failure at $1-\alpha$ level.

5. $\hat{\mu}_{k,N_1+k}^{(2i-1)} \leftarrow \frac{1}{N_1} \sum_{j=k+1}^{N_1+k} X_j^{(2i-1)}$ $\quad i = 1,\ldots,m'$.

6. $\hat{\mu}_{k,N_2+k}^{(2i)} \leftarrow \frac{1}{N_2} \sum_{j=k+1}^{N_2+k} X_j^{(2i)}$ $\quad i = 1,\ldots,m'$.

7. $A \leftarrow \sum_{i=1}^{m'} \hat{\mu}_{k,N_1+k}^{(2i-1)}$.

8. $B \leftarrow \sum_{i=1}^{m'} \hat{\mu}_{k,N_2+k}^{(2i)}$.

9. $\tilde{\mu} \leftarrow (N_2B - N_1A)/m'(N_2-N_1)$.

10. $\tilde{g}_k \leftarrow N_1N_2\ (A-B)/m'(N_2-N_1)$.

11. $\hat{\sigma}^2 \leftarrow \frac{1}{m-2} \sum_{i=1}^{m'} [N_1(\hat{\mu}_{k,N_1+k}^{(2i-1)} - A/m')^2 + N_2(\hat{\mu}_{k,N_2+k}^{(2i)} - B/m')^2]$.

Test: $g_k/N_1\mu = 0$.

12. $h \leftarrow N_1N_2\hat{\sigma}^2 [2\ t_{m-2}(\alpha)/m(N_2-N_1)]^2$.

13. $\gamma^2 \leftarrow (\tilde{\mu}\ \tilde{g}_k - hm)^2 - \left[\tilde{\mu}^2 - \frac{hm(N_1+N_2)}{2N_1N_2}\right] \cdot \left[\tilde{g}_k^2 - \frac{hm(N_1+N_2)}{2}\right]$.

14. If $\gamma^2 < 0$ go to 18 .

15. $r_1 \leftarrow \frac{\tilde{\mu}\ \tilde{g}_k - hm - \gamma}{\tilde{\mu}^2 - hm(N_1+N_2)/2N_1N_2}$.

16. $r_2 \leftarrow \frac{\tilde{\mu}\ \tilde{g}_k - hm + \gamma}{\tilde{\mu}^2 - hm(N_1+N_2)/2N_1N_2}$.

17. If $\max(|r_1|, |r_2|) \leq \delta$ go to 23 .

Relative bias is significant at $1-\alpha$ level.

18. $\hat{var}(\tilde{\mu}) \leftarrow \hat{\sigma}^2 (N_1+N_2)/m(N_2-N_1)^2$.

19. $\hat{var}(\tilde{g}_k) \leftarrow 2N_1N_2\ \hat{var}(\tilde{\mu})$.

20. $I_1 \leftarrow \tilde{\mu} - t_{m-2}(\alpha)\sqrt{\hat{var}(\tilde{\mu})}$.

21. $I_2 \leftarrow -I_1 + 2\tilde{\mu}$.

22. Return with

point estimate $\tilde{\mu}$,

$1 - \alpha$ interval estimate $[I_1, I_2]$,

bias estimate $\tilde{g}_k$,

variances $\hat{var}(\tilde{\mu})$ and $\hat{var}(\tilde{g}_k)$.

Relative bias is not significant at $1-\alpha$ level.

23. $$\hat{\mu} \leftarrow \frac{\sqrt{N_1}\,A + \sqrt{N_2}\,B}{m'(\sqrt{N_1} + \sqrt{N_2})} .$$

24. $\hat{var}(\hat{\mu}) \leftarrow 4\hat{\sigma}^2/m\,(\sqrt{N_1} + \sqrt{N_2})^2$.

25. $I_1 \leftarrow \hat{\mu} - t_{m-2}(\alpha)\sqrt{\hat{var}(\hat{\mu})}$.

26. $I_2 \leftarrow -I_1 + 2\hat{\mu}$.

27. Return with

point estimate $\hat{\mu}$,

$1 - \alpha$ interval estimate $[I_1, I_2]$,

variance $\hat{var}(\hat{\mu})$,

indication that $g_k/N_1\mu = 0$ at $1-\alpha$ level.

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This paper presents a perspective on the initial transient problem in steady-state simulation. In particular, it enumerates five generally accepted facts: 1) Conditions prevailing at the beginning of a simulation influence sample paths. 2) The extent of influence is a function of the strength of autocorrelation. 3) Some initial conditions are less detrimental than others are. 4) Truncation reduces bias but usually increases variance. 5) So far no complete solution exists. The remainder

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20. of the paper describes a proposal for solving the problem. It relies on the relatively weak assumption that the conditional means in a stochastic process of interest are related linearly. An estimator of the steady-state mean is described which has considerably less bias than one can achieve via conventional truncation. An interval estimator is also described which follows from standard regression theory. A test for residual bias is presented which enables a user to judge whether or not sample data meet the minimal requirements for the proposed technique to apply. A second test allows a user to judge whether or not a more efficient estimation technique can be used.

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